SECTION 7.7 EXERCISES

Review Questions

1. What are the two general ways in which an improper integral may occur?
2. Explain how to evaluate $\int_a^\infty f(x)\,dx$.
3. Explain how to evaluate $\int_0^1 x^{-1/2}\,dx$.
4. For what values of p does $\int_1^\infty x^{-p}\,dx$ converge?

Basic Skills

5–20. Infinite intervals of integration *Evaluate the following integrals or state that they diverge.*

5. $\displaystyle\int_1^\infty x^{-2}\,dx$

6. $\displaystyle\int_0^\infty \frac{dx}{(x+1)^3}$

7. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}}$

8. $\displaystyle\int_0^\infty \frac{dx}{\sqrt[3]{x+2}}$

9. $\displaystyle\int_0^\infty e^{-2x}\,dx$

10. $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$

11. $\displaystyle\int_{e^2}^\infty \frac{dx}{x\ln^p x},\ p > 1$

12. $\displaystyle\int_0^\infty \frac{x}{\sqrt[5]{x^2+1}}\,dx$

13. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

14. $\displaystyle\int_0^\infty \cos x\,dx$

15. $\displaystyle\int_2^\infty \frac{\cos(\pi/x)}{x^2}\,dx$

16. $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$

17. $\displaystyle\int_0^\infty \frac{x}{\sqrt{x^4+1}}\,dx$

18. $\displaystyle\int_a^\infty \sqrt{e^{-x}}\,dx,$ for any finite constant a

19. $\displaystyle\int_2^\infty \frac{x}{(x+2)^2}\,dx$

20. $\displaystyle\int_1^\infty \frac{\tan^{-1}x}{x^2+1}\,dx$

21–26. Volumes on infinite intervals *Find the volume of the described solid of revolution or state that it does not exist.*

21. The region bounded by $f(x) = x^{-2}$ and the x-axis on the interval $[1, \infty)$ is revolved about the x-axis.
22. The region bounded by $f(x) = (x^2+1)^{-1/2}$ and the x-axis on the interval $[2, \infty)$ is revolved about the x-axis.
23. The region bounded by $f(x) = \sqrt{\dfrac{x+1}{x^3}}$ and the x-axis on the interval $[1, \infty)$ is revolved about the x-axis.
24. The region bounded by $f(x) = (x+1)^{-3}$ and the x-axis on the interval $[0, \infty)$ is revolved about the y-axis.
25. The region bounded by $f(x) = \dfrac{1}{\sqrt{x}\ln x}$ and the x-axis on the interval $[2, \infty)$ is revolved about the x-axis.
26. The region bounded by $f(x) = \dfrac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the x-axis on the interval $[0, \infty)$ is revolved about the x-axis.

27–36. Integrals with unbounded integrands *Evaluate the following integrals or state that they diverge.*

27. $\displaystyle\int_0^8 \frac{dx}{\sqrt[3]{x}}$

28. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

29. $\displaystyle\int_0^1 \frac{x^3}{x^4-1}\,dx$

30. $\displaystyle\int_1^\infty \frac{dx}{\sqrt[3]{x-1}}$

31. $\displaystyle\int_0^{10} \frac{dx}{\sqrt[4]{10-x}}$

32. $\displaystyle\int_1^{11} \frac{dx}{(x-3)^{2/3}}$

33. $\displaystyle\int_0^1 \ln x^2\,dx$

34. $\displaystyle\int_{-1}^1 \frac{x}{x^2+2x+1}\,dx$

35. $\displaystyle\int_{-2}^2 \frac{dx}{\sqrt{4-x^2}}$

36. $\displaystyle\int_0^{\pi/2} \sec\theta\,d\theta$

37–40. Volumes with infinite integrands *Find the volume of the described solid of revolution or state that it does not exist.*

37. The region bounded by $f(x) = (x-1)^{-1/4}$ and the x-axis on the interval $(1, 2]$ is revolved about the x-axis.
38. The region bounded by $f(x) = (x^2-1)^{-1/4}$ and the x-axis on the interval $(1, 2]$ is revolved about the y-axis.
39. The region bounded by $f(x) = (4-x)^{-1/3}$ and the x-axis on the interval $[0, 4)$ is revolved about the y-axis.
40. The region bounded by $f(x) = (x+1)^{-3/2}$ and the y-axis on the interval $(-1, 1]$ is revolved about the line $x = -1$.
41. **Arc length** Find the length of the hypocycloid (or astroid) $x^{2/3} + y^{2/3} = 4$.
42. **Circumference of a circle** Use calculus to find the circumference of a circle with radius a.
43. **Bioavailability** When a drug is given intravenously, the concentration of the drug in the blood is $C_i(t) = 250e^{-0.08t}$, for $t \ge 0$. When the same drug is given orally, the concentration of the drug in the blood is $C_o(t) = 200(e^{-0.08t} - e^{-1.8t})$, for $t \ge 0$. Compute the bioavailability of the drug.
44. **Draining a pool** Water is drained from a swimming pool at a rate given by $R(t) = 100e^{-0.05t}$ gal/hr. If the drain is left open indefinitely, how much water is drained from the pool?
45. **Maximum distance** An object moves on a line with velocity $v(t) = 10/(t+1)^2$ mi/hr for $t \ge 0$. What is the maximum distance the object can travel?

46. **Depletion of oil reserves** Suppose that the rate at which a company extracts oil is given by $r(t) = r_0e^{-kt}$, where $r_0 = 10^7$ barrels/yr and $k = 0.005\ \text{yr}^{-1}$. Suppose also the estimate of the total oil reserve is 2×10^9 barrels. If the extraction continues indefinitely, will the reserve be exhausted?

Further Explorations

47. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
    a. If f is continuous and $0 < f(x) < g(x)$ on the interval $[0, \infty)$ and $\int_0^\infty g(x)\,dx = M < \infty$, then $\int_0^\infty f(x)\,dx$ exists.
    b. If $\lim_{x\to\infty} f(x) = 1$, then $\int_0^\infty f(x)\,dx$ exists.
    c. If $\int_0^1 x^{-p}\,dx$ exists, then $\int_0^1 x^{-q}\,dx$ exists, where $q > p$.
    d. If $\int_1^\infty x^{-p}\,dx$ exists, then $\int_1^\infty x^{-q}\,dx$ exists, where $q > p$.
    e. $\displaystyle\int_1^\infty \frac{dx}{x^{3p+2}}$ exists for $p > -\frac{1}{3}$.
48. **Incorrect calculation** What is wrong with this calculation?
$$\int_{-1}^1 \frac{dx}{x} = \ln|x|\Big|_{-1}^1 = \ln 1 - \ln 1 = 0$$
49. **Using symmetry** Use symmetry to evaluate the following integrals.
    a. $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx$
    b. $\displaystyle\int_{-\infty}^\infty \frac{x^3}{1+x^8}\,dx$
50. **Integral with a parameter** For what values of p does the integral $\displaystyle\int_2^\infty \frac{dx}{x\ln^p x}$ exist and what is its value (in terms of p)?
51. **Improper integrals by numerical methods** Use the Trapezoid Rule (Section 7.6) to approximate $\int_0^R e^{-x^2}\,dx$ with $R = 2, 4,$ and 8. For each value of R, take $n = 4, 8, 16,$ and 32, and compare approximations with successive values of n. Use these approximations to approximate $I = \int_0^\infty e^{-x^2}\,dx$.

52–54. Integration by parts *Use integration by parts to evaluate the following improper integrals.*

52. $\displaystyle\int_0^\infty xe^{-x}\,dx$

53. $\displaystyle\int_0^1 x\ln x\,dx$

54. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

55. **A close comparison** Graph the integrands; then, evaluate and compare the values of $\int_0^\infty xe^{-x^2}\,dx$ and $\int_0^\infty x^2e^{-x^2}\,dx$.
56. **Area between curves** Let R be the region bounded by the graphs of $y = x^{-p}$ and $y = x^{-q}$ for $x \ge 1$, where $q > p > 1$. Find the area of R.
57. **Area between curves** Let R be the region bounded by the graphs of $y = e^{-ax}$ and $y = e^{-bx}$ for $x \ge 0$, where $a > b > 0$. Find the area of R.
58. **An area function** Let $A(a)$ denote the area of the region bounded by $y = e^{-ax}$ and the x-axis on the interval $[0, \infty)$. Graph the function $A(a)$ for $0 < a < \infty$. Describe how the area of the region decreases as the parameter a increases.
59. **Regions bounded by exponentials** Let $a > 0$ and let R be the region bounded by the graph of $y = e^{-ax}$ and the x-axis on the interval $[b, \infty)$.
    a. Find $A(a, b)$, the area of R as a function of a and b.
    b. Find the relationship $b = g(a)$ such that $A(a, b) = 2$.
    c. What is the minimum value of b (call it b^*) such that when $b > b^*$, $A(a, b) = 2$ for some value of $a > 0$?
60. **The family $f(x) = 1/x^p$ revisited** Consider the family of functions $f(x) = 1/x^p$, where p is a real number. For what values of p does the integral $\int_0^1 f(x)\,dx$ exist? What is its value?
61. **When is the volume finite?** Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x-axis for $0 < x < 1$.
    a. Let S be the solid generated when R is revolved about the x-axis. For what values of p is the volume of S finite?
    b. Let S be the solid generated when R is revolved about the y-axis. For what values of p is the volume of S finite?
62. **When is the volume finite?** Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x-axis for $x \ge 1$.
    a. Let S be the solid generated when R is revolved about the x-axis. For what values of p is the volume of S finite?
    b. Let S be the solid generated when R is revolved about the y-axis. For what values of p is the volume of S finite?

63–66. By all means *Use any means to verify (or approximate as closely as possible) the following integrals.*

63. $\displaystyle\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln(\cos x)\,dx = -\frac{\pi\ln 2}{2}$

64. $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

65. $\displaystyle\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}$

66. $\displaystyle\int_0^1 \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$

Applications

67. **Perpetual annuity** Imagine that today you deposit $\$B$ in a savings account that earns interest at a rate of $p\%$ per year compounded continuously (see Section 6.8). The goal is to draw an income of $\$I$ per year from the account forever. The amount of money that must be deposited is $B = I\int_0^\infty e^{-rt}\,dt$, where $r = p/100$. Suppose you find an account that earns 12% interest annually and you wish to have an income from the account of \$5000 per year. How much must you deposit today?
68. **Draining a tank** Water is drained from a 3000-gal tank at a rate that starts at 100 gal/hr and decreases continuously by 5%/hr. If the drain is left open indefinitely, how much water is drained from the tank? Can a full tank be emptied at this rate?
69. **Decaying oscillations** Let $a > 0$ and b be real numbers. Use integration to confirm the following identities.
    a. $\displaystyle\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$
    b. $\displaystyle\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

70. Electronic chips Suppose the probability that a particular computer chip fails after $t = a$ hours of operation is $0.00005\int_a^\infty e^{-0.00005t}\,dt$.

a. Find the probability that the computer chip fails after 15,000 hr of operation.

b. Of the chips that are still operating after 15,000 hr, what fraction of these will operate for at least another 15,000 hr?

c. Evaluate $0.00005\int_0^\infty e^{-0.00005t}\,dt$ and interpret its meaning.

71. Average lifetime The average time until a computer chip fails (see Exercise 70) is $0.00005\int_0^\infty te^{-0.00005t}\,dt$. Find this value.

72. The Eiffel Tower property Let R be the region between the curves $y = e^{-cx}$ and $y = -e^{-cx}$ on the interval $[a, \infty)$, where $a \geq 0$ and $c > 0$. The center of mass of R is located at $(\bar{x}, 0)$, where $\bar{x} = \dfrac{\int_a^\infty xe^{-cx}\,dx}{\int_a^\infty e^{-cx}\,dx}$. (The profile of the Eiffel Tower is modeled by the two exponential curves.)

a. For $a = 0$ and $c = 2$, sketch the curves that define R and find the center of mass of R. Indicate the location of the center of mass.

b. With $a = 0$ and $c = 2$, find equations of the tangent lines to the curves at the points corresponding to $x = 0$.

c. Show that the tangent lines intersect at the center of mass.

d. Show that this same property holds for any $a \geq 0$ and any $c > 0$; that is, the tangent lines to the curves $y = \pm e^{-cx}$ at $x = a$ intersect at the center of mass of R.

(*Source:* P. Weidman and I. Pinelis, *Comptes Rendu, Mechanique* 332 (2004): 571–584. Also see the Guided Projects.)

73. Escape velocity and black holes The work required to launch an object from the surface of Earth to outer space is given by $W = \int_R^\infty F(x)\,dx$, where $R = 6370$ km is the approximate radius of Earth, $F(x) = GMm/x^2$ is the gravitational force between Earth and the object, G is the gravitational constant, M is the mass of Earth, m is the mass of the object, and $GM = 4 \times 10^{14}\,\text{m}^3/\text{s}^2$.

a. Find the work required to launch an object in terms of m.

b. What escape velocity v_e is required to give the object a kinetic energy $\frac{1}{2}mv_e^2$ equal to W?

c. The French scientist Laplace anticipated the existence of black holes in the 18th century with the following argument: If a body has an escape velocity that equals or exceeds the speed of light, $c = 300{,}000$ km/s, then light cannot escape the body and it cannot be seen. Show that such a body has a radius $R \leq 2GM/c^2$. For Earth to be a black hole, what would its radius need to be?

74. Adding a proton to a nucleus The nucleus of an atom is positively charged because it consists of positively charged protons and uncharged neutrons. To bring a free proton toward a nucleus, a repulsive force $F(r) = kqQ/r^2$ must be overcome, where $q = 1.6 \times 10^{-19}$ C is the charge on the proton, $k = 9 \times 10^9\,\text{N-m}^2/\text{C}^2$, Q is the charge on the nucleus, and r is the distance between the center of the nucleus and the proton. Find the work required to bring a free proton (assumed to be a point mass) from a large distance $(r \to \infty)$ to the edge of a nucleus that has a charge $Q = 50q$ and a radius of 6×10^{-11} m.

T **75. Gaussians** An important function in statistics is the Gaussian (or normal distribution, or bell-shaped curve), $f(x) = e^{-ax^2}$.

a. Graph the Gaussian for $a = 0.5, 1$, and 2.

b. Given that $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\dfrac{\pi}{a}}$, compute the area under the curves in part (a).

c. Complete the square to evaluate $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx$, where $a > 0$, b, and c are real numbers.

76–80. Laplace transforms *A powerful tool in solving problems in engineering and physics is the Laplace transform. Given a function $f(t)$, the Laplace transform is a new function $F(s)$ defined by*

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

where we assume that s is a positive real number. For example, to find the Laplace transform of $f(t) = e^{-t}$, the following improper integral is evaluated using integration by parts:

$$F(s) = \int_0^\infty e^{-st}e^{-t}\,dt = \int_0^\infty e^{-(s+1)t}\,dt = \frac{1}{s+1}$$

Verify the following Laplace transforms, where a is a real number.

76. $f(t) = 1 \;\rightarrow\; F(s) = \dfrac{1}{s}$

77. $f(t) = e^{at} \;\rightarrow\; F(s) = \dfrac{1}{s-a}$

78. $f(t) = t \;\rightarrow\; F(s) = \dfrac{1}{s^2}$

79. $f(t) = \sin at \;\rightarrow\; F(s) = \dfrac{a}{s^2 + a^2}$

80. $f(t) = \cos at \;\rightarrow\; F(s) = \dfrac{s}{s^2 + a^2}$

Additional Exercises

81. Improper integrals Evaluate the following improper integrals (Putnam Exam, 1939).

a. $\displaystyle\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}}$ b. $\displaystyle\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}}$

82. A better way Compute $\int_0^1 \ln x\,dx$ using integration by parts. Then explain why $-\int_0^\infty e^{-x}\,dx$ (an easier integral) gives the same result.

83. Competing powers For what values of $p > 0$ is $\displaystyle\int_0^\infty \frac{dx}{x^p + x^{-p}} < \infty$?

84. Gamma function The gamma function is defined by $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx$, for p not equal to zero or a negative integer.

a. Use the reduction formula

$$\int_0^\infty x^p e^{-x}\,dx = p\int_0^\infty x^{p-1}e^{-x}\,dx \quad \text{for } p = 1, 2, 3, \ldots$$

to show that $\Gamma(p+1) = p!$ (p factorial).

b. Use the substitution $x = u^2$ and the fact that $\displaystyle\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$ to show that $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.

85. Many methods needed Show that $\displaystyle\int_0^\infty \frac{\sqrt{x}\ln x}{(1+x)^2}\,dx = \pi$ in the following steps.

a. Integrate by parts with $u = \sqrt{x}\ln x$.

b. Change variables by letting $y = 1/x$.

c. Show that $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}(1+x)}\,dx = -\int_1^\infty \frac{\ln x}{\sqrt{x}(1+x)}\,dx$ and conclude that $\displaystyle\int_0^\infty \frac{\ln x}{\sqrt{x}(1+x)}\,dx = 0$.

d. Evaluate the remaining integral using the change of variables $z = \sqrt{x}$.

(*Source: Mathematics Magazine* 59, no. 1, February 1986: 49)

86. Riemann sums to integrals Show that $L = \displaystyle\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right) = -1$ in the following steps.

a. Note that $n! = n(n-1)(n-2)\cdots 1$ and use $\ln(ab) = \ln a + \ln b$ to show that

$$L = \lim_{n\to\infty}\left[\left(\frac{1}{n}\sum_{k=1}^{n}\ln k\right) - \ln n\right] = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right)$$

b. Identify the limit of this sum as a Riemann sum for $\int_0^1 \ln x\,dx$. Integrate this improper integral by parts and reach the desired conclusion.

QUICK CHECK ANSWERS

1. The integral diverges. $\displaystyle\lim_{b\to\infty}\int_1^b (1 + x^{-1})\,dx = \lim_{b\to\infty}(x + \ln x)\Big|_1^b$ does not exist. 2. $\frac{1}{3}$ 3. c must approach 0 through values in the interval of integration (0, 1). Therefore, $c \to 0^+$. ◂

7.8 Introduction to Differential Equations

▸ If you read Sections 4.8 and 6.1, then you have already encountered a preview of differential equations. Given the derivative of a function, these two sections show how to find the function itself by integration. This process amounts to solving a differential equation.

If you had to demonstrate the utility of mathematics to a skeptic, a convincing way would be to cite *differential equations*. This vast subject lies at the heart of mathematical modeling and is used in engineering, the natural and biological sciences, economics, management, and finance. Differential equations rely heavily on calculus, and they are usually studied in advanced courses that follow calculus. Nevertheless, you have now seen enough calculus to understand a brief survey of differential equations and appreciate their power.

An Overview

A differential equation involves an unknown function y and its derivatives. The unknown in a differential equation is not a number (as in an algebraic equation), but rather a function or a relationship. Here are some examples of differential equations:

$$\text{(A)}\ \frac{dy}{dx} + 4y = \cos x \qquad \text{(B)}\ \frac{d^2y}{dx^2} + 16y = 0 \qquad \text{(C)}\ y'(t) = 0.1y(100 - y)$$

In each case, the goal is to find a function y that satisfies the equation.

▸ Common choices for the independent variable in a differentiable equation are x and t, with t being used for time-dependent problems.

▸ A *linear* differential equation cannot have terms such as y^2, yy', or $\sin y$, where y is the unknown function. The most general first-order linear equation has the form $y' + py = q$, where p and q are functions of the independent variable.

The **order** of a differential equation is the order of the highest-order derivative that appears in the equation. Of the three differential equations just given, (A) and (C) are first order, and (B) is second order. A differential equation is **linear** if the unknown function and its derivatives appear only to the first power and are not composed with other functions. Of these equations, (A) and (B) are linear, but (C) is nonlinear (because the right side contains y^2).

Solving a first-order differential equation requires integration—you must "undo" the derivative $y'(t)$ in order to find $y(t)$. Integration introduces an arbitrary constant, so the **general solution** of a first-order differential equation involves one arbitrary constant. Similarly, the general solution of a second-order differential equation involves two arbitrary constants; with an nth-order differential equation, the general solution involves n arbitrary constants.